Examples for integral calculus

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1. Find $\int_0^2 x^2\,dx$, $\int_{-2}^2 \left(4-x^2\right)\,dx$ and $\int_{-1}^1 \frac{1}{x^3}dx$.

Solution.

$$\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

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$$\int_{-2}^2 \left(4-x^2\right)\,dx = \left(4x - \left.\frac{x^3}{3}\right|_{-2}^2 = \frac{32}{3}\right.$$

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Consider $\int_{-1}^1 \frac{1}{x^3}dx = F(1) - F(-1)$. Here $F(1) = 1$ and $F(-1) = -1$ But $\infty = F(0^+) \neq F(0^-) = -\infty$, which means that $f(x) = 1/x^3$ is not continuous at 0, which is a point in $[-1, 1]$. Therefore the definite integral given does not exist.

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2. Find $\int \frac{2z}{\sqrt[3]{z^2+1}}dz$

Solution. (1) We know that if u is a differentiable function of x, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

provided that $n \neq -1$. Let $u = z^2 + 1$, then $du = 2z\,dz$. And then,

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2+1}} &= \int \frac{1}{u^{\frac{1}{3}}}du \\ &= \int u^{-\frac{1}{3}}\,du \\ &= \frac{3}{2}u^{\frac{2}{3}} + C \\ &= \frac{3}{2}\left(z^2+1\right)^{\frac{2}{3}} + C\end{aligned}$$

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Solution. (2) Let $u = \sqrt[3]{z^2+1}$, then $3u^2\,du = 2z\,dz$.

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2+1}}dz &= \int \frac{3u^2}{u}du \\ &= 3\int u\,du \\ &= 3\frac{u^2}{2} + C \\ &= \frac{3}{2}\left(z^2+1\right)^{\frac{3}{2}} + C\end{aligned}$$

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3. Evaluate $\int_{-1}^1 3x^2\sqrt{x^3+1}\,dx$.

Solution. (1) Let $u = x^3 + 1$. Then $du = 3x^2 dx$, $u(-1) = 0$ and $u(1) = 2$. And then,

$$\begin{aligned}\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx &= \int_0^2 \sqrt{u}du \\ &= \frac{2}{3}u^{\frac{3}{2}}\Big|_0^2 \\ &= \frac{4\sqrt{2}}{3}\end{aligned}$$

Solution. (2) Again, substitute the u and du as above. Then find the indefinite integral,

$$\begin{aligned}\int 3x^2\sqrt{x^2+1}\,dx &= \int \sqrt{u}du \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}} + C\end{aligned}$$

And then consider the corresponding definite integral,

$$\begin{aligned}\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx &= \frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}}\Big|_{-1}^{1} \\ &= \frac{4\sqrt{2}}{3}\end{aligned}$$

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4. Find the average value of $f(x) = \sqrt{4-x^2}dx$ on the interval $[-2, 2]$.

Solution. The average value required is,

$$\frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\,dx = \frac{\pi}{2}$$

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Bibliography

George B Thomas, Jr and Ross L Finney. *Calculus and analytic geometry.* 8th, 1992